

Singular thin viscous sheet

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The evolution of a thin viscous layer is usually smooth. Here we conduct an experiment where the layer adopts a singular shape. Using the analogy between the flow of a viscous liquid and the deformation of an elastic solid, the theoretical analysis predicts a conical shape for the sheet and is in quantitative agreement with the experiment.

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In numerous coating and casting operations, a liquid layer is removed from a roller and coated onto a web or other solid substrate. Product quality control requirements typically place constraints on the planarity and uniformity of the free surface of these films. These coating processes can be found in fiber coating [1], nanotechnology and electronics [2], pharmacology [3], solid mechanics testing [4], and optical devices and panels [5]. Thin layers of viscous liquids [6–14] have received much attention during the past few years. In these studies, the viscous sheets buckle or fold smoothly. Here we show that a viscous sheet may adopt a singular shape.

The work on viscous fluids in slender geometries have been stimulated by Taylor [15], who recalled an analogy between the flow of a viscous liquid and the deformations of an elastic solid. A thin elastic plate is similar to a thin viscous sheet. Thin elastic films, when deformed, suffer two types of deformations: stretching and bending [16]. It is easy to bend a thin elastic sheet and it is difficult to stretch it. Energetically speaking, stretching is much more expensive than bending. As a consequence, large deformations of thin elastic sheets lead to the formation of singular structures like the one observed in a crumpled paper; there is pure bending almost everywhere and the expensive stretching is localized near the singularities [17–19]. In this paper, we observe conical singularities in viscous sheets similar to the one found in elastic sheets [20–24].

First, let us explain the elastic-viscous analogy. The stress tensor for an incompressible [25] elastic solid of Young's modulus E , $\sigma_{ij} = -p\delta_{ij} + 2E/3u_{ij}$, has the same form as for an incompressible newtonian fluid of viscosity μ , $\sigma_{ij} = -p\delta_{ij} + 2\mu du_{ij}/dt$, p being the pressure, δ_{ij} the identity tensor, and u_{ij} the strain tensor. If τ is a characteristic time of the flow, the equivalent of Young's modulus becomes $3\mu/\tau$. The analogy results from the identity between the equations describing the solid equilibrium $\partial_i\sigma_{ij} = 0$ and the Stokes equation describing a small Reynolds number flow. In other words, the minimization of dissipation in a viscous flow is mathematically equivalent to the minimization of elastic energy in a solid. For thin incompressible elastic (viscous) sheets of thickness h and typical size R ($h \ll R$), a bending deformation has a typical energy $\kappa = Eh^3/9$ (dissipation $\mu h^3/3$), κ being the bending modulus; a stretching (shearing) deformation has an energy $\kappa(R/h)^2$ (dissipation $\mu hR^2/3$). As a consequence, in both cases, bending deformations are preferred. Equivalently [12], the time scale for stretching is larger than for bending by a factor $(R/h)^2 \gg 1$.

To observe conical singularities for viscous sheets, we have devised an experiment showing the viscous analog of a circular cloth held on a stick. The experiment consists in filling a cylindrical tank with water and preparing a circular sheet of silicon oil floating on the water surface. The silicon oil is less dense than water and 600 000 times more viscous. A stick is placed along the cylinder axis and its tip is under the water surface. The water is drained from the bottom of the cylinder. The silicon oil layer falls down with the water and when it reaches the stick it displays the equivalent of a cloth.

The radius R of the sheet is a few centimeters, while its thickness h is of the order of the capillary length $\sqrt{\gamma/\rho g} = 1.5$ mm, g being the acceleration of gravity (other quantities are defined in Fig. 1). The water is drained through a quick opening tap, a fast steady flow is obtained for 2 s, which is more than the typical time 1 s for an experiment. The water surface is dropping at a rate $v = 7.6$ cm/s.

When the viscous sheet reaches the stick, a small circular part of radius $R_c \sim 2$ mm of the sheet adheres to the stick (Fig. 1). It is the equivalent of the core of a d-cone. 0.5 s later, the rest of the sheet folds in the orthoradial direction (Fig. 1) and air penetrates under the sheet. The shape of the sheet resembles a cone with the apex located at the tip of the stick. Later on (1 s), a few folds grow larger while the others disappear. Eventually, the sheet is all stuck to the stick along which the fluid flows with a time scale of many minutes. We have measured the number of folds just after their formation (Fig. 3). It increases with the ratio R/h .

More insight can be gained with the modeling of the experiment. We first consider the elastic equivalent problem: A circular sheet placed on a stick, in the gravitation field. Due to the high cost of stretching and the stick constraint, we expect a conical shape to develop so that, in polar coordinates (r, θ) , the vertical deflection of the sheet is given by $\xi = rf(\theta)$. The shape f is given by the minimization of

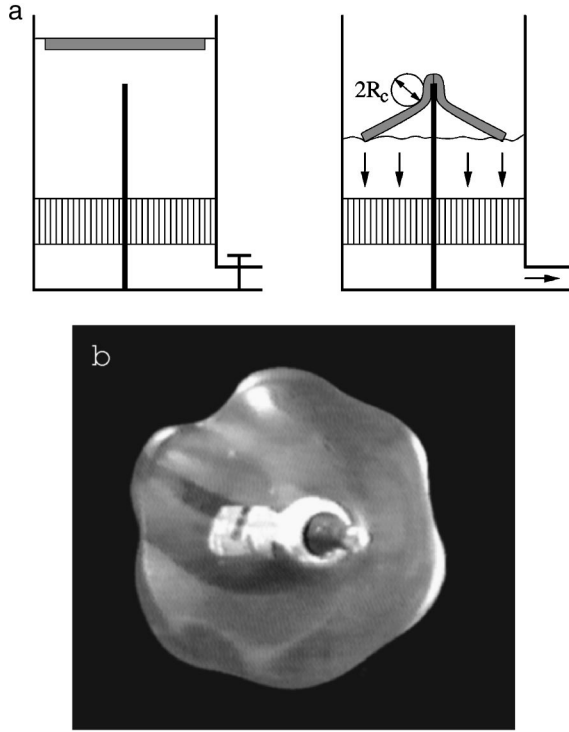


FIG. 1. Top: Scheme of the experiment. A circular thin layer of silicon oil (mass density $\rho=980 \text{ kg/m}^3$, viscosity $\mu=600 \text{ Pa}\cdot\text{s}$, surface tension $\gamma=21 \text{ mN/m}$) is prepared on top of water. The water is drained from the bottom, passing through a network of straws filling the section of the container to achieve a laminar flow. The velocity of the surface of the water is $v=7.6 \text{ cm/s}$. When the sheet reaches the stick, a small zone near its center adheres to the stick, air penetrates under the sheet and only its periphery remains in contact with water. A periodic pattern develops then. The thickness, the vertical scale and the zone near the stick are magnified. The radius of the core R_c is defined here. Bottom: Top view of the experiment, the camera has been slightly inclined. The stick appears under the sheet. The periodic pattern has just formed, with a wave number $n=6$. The viscous layer has a thickness $h=2.4 \text{ mm}$ and a radius $R=3.6 \text{ cm}$.

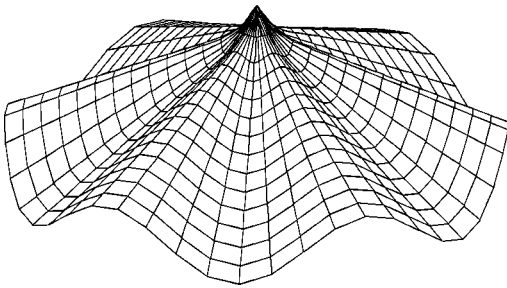


FIG. 2. The theoretical shape of the viscous sheet showing a periodic pattern with a wave number $n=6$, corresponding to $\eta=36$, η being the number comparing gravitational and bending energies [defined by Eq. (2)]. The center of the sheet has been corrected to show the core of the conical singularity. This shape is similar to that of a circular cloth put on a stick.

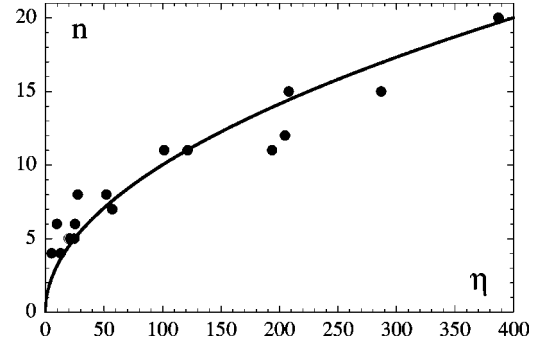


FIG. 3. The orthoradial wave number of the periodic pattern as a function of the nondimensional parameter $\eta = (\rho g R^4)/(\mu h^2 v \ln(R/h))$ (Eq. 2), which compares gravitational and bending energy. The circles are experimental points and the continuous line corresponds to $n = \sqrt{\eta}$ [Eq. (3)].

$$\mathcal{M} = \frac{1}{2} \kappa \ln\left(\frac{R}{R_c}\right) \int (f + f'')^2 d\theta - \frac{1}{3} \rho g h R^3 \int f d\theta + \frac{1}{2} \lambda \int f(f + f'') d\theta. \quad (1)$$

The first term is the bending energy, proportional to the bending modulus κ , the second is the gravitational energy, and the third enforces the constraint of no stretching [19,21] using a Lagrange multiplier λ . The first and last terms are correct for not too large f . The gravitational energy is relevant as for cloths [26]. The solutions (up to rotations) are $f(\theta) = \eta n^{-2} (1 + \sqrt{2/(n^2-1)} \sin(n\theta))$, the orthoradial (azimuthal) wave number n is integer and $\eta = (R^3 \rho g h)/(3 \kappa \ln(R/R_c))$ is the nondimensional parameter comparing gravitational and elastic effects. Thus, a cloth on a stick shows a periodic pattern as can be observed in Fig. 2. The wave number n is selected by the boundary conditions.

Let us turn now to the viscous transposition. The time scale τ is defined by the experimental conditions and is imposed here by the boundary conditions. The periphery of the sheet of radius R is sustained by the water and falling at a velocity v . As a consequence, $\tau = R/v$. Also, it has to be kept in mind that this time scale is much smaller than the time scale of the flow inside the sheet [their ratio is $(h/R)^2 \sim 10^{-4}$]. The equivalent of the bending modulus $Eh^3/9$ of an elastic sheet is then $\kappa = \mu h^3/(3\tau)$. As the sheet central zone adheres to the stick, the size of the core is not the same as in the elastic case. However, it can be given by the same arguments: The bending E_b and the stretching E_s energy have the same order of magnitude (in elastic terms). In the core, the typical strain is $v_c \tau/R_c$ and the curvature is $1/R_c$, v_c being the typical velocity in the core. To match to the conical shape outside the core, $v_c/R_c \sim v/R$. As a consequence, $E_b \sim E_s$ leads to $R_c \sim h$; the proportionality factor is unimportant because the dependence of η on R_c is only logarithmic. To sum up, the nondimensional parameter governing the problem reads

$$\eta = \frac{\rho g R^4}{\mu h^2 v \ln(R/h)}. \quad (2)$$

It is a nontrivial function of almost all the parameters of the system. As in the elastic case, the wave number is selected by the boundary conditions. Here, the sheet is sustained by the water: The mean velocity of the periphery of the sheet is v , i.e., $\langle d\xi/dt(R, \theta) \rangle = R \int f(\theta) d\theta / \tau = \eta R / (n^2 \tau) = v$. As a consequence,

$$n = \sqrt{\eta}. \quad (3)$$

We could expect that n increases with η . For instance, at small η , gravity is unimportant, the system behaves as an elastic sheet [22,23], for which the orthoradial wave number

is 1. The comparison with the experimental data (Fig. 3) shows good agreement with no adjustable parameter. Thus, within an experimental time scale, the viscous sheet adopts a conical shape similar to a conical singularity in an elastic sheet. This conical figure, would give the viscous sheet its final shape when coating the stick.

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